Problem Solving 2018/19

Week 8

1. Prove that $2^{6n} + 3^{2n-2}$ is divisible by 5 for all $n \geq 1$.

   (UCC Maths Enrichment)

2. Let G be a graph with an odd number of vertices. Let the degree of a vertex be the number of edges incident to it (the number of edges that start/end at it). Prove that there is a vertex with even degree.

3. Let $x_1, \ldots, x_n \geq 0$. Prove that

$$(x_1 + \cdots + x_n)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right) \geq n^2.$$

   (UCC Maths Enrichment)

4. Find all positive integers n for which $n^8 + n + 1$ is prime.

   (UCC Maths Enrichment)

5. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$2f(x) = f(x+y) + f(x+2y)$$

   for all $x \in \mathbb{R}$ and $y \geq 0$.

   (Romanian IMO Team Selection Test 2011)

6. As we all know, the Hamilton, the Dining Hall, the Arts Block, the GMB, the Museum Building and the Berkeley are connected by underground tunnels. These tunnels were built in such a way that there is exactly one tunnel directly connecting each pair of the above buildings. Each tunnel is built either entirely of gold, or entirely of silver. Prove that there are four buildings for which you can go on a round trip visiting all of them through tunnels of the same material. (Note that a round trip of four buildings P, Q, R and S, is a journey that follows the path $P \to Q \to R \to S \to P$.)

   (IrMO 2007 P1 Q4) (Edited, maybe)

7. Emerald writes 2009^2 integers in a 2009×2009 table, one number in each cell. She sums all the numbers in each row and in each column, obtaining 4018 sums. She notices that all sums are distinct. Is it possible that all such sums are perfect squares?

(Brazilian Mathematical Olympiad 2009 P1 Q1)

8. Suppose A, B, C, D are $n \times n$ matrices, satisfying the conditions that AB^t and CD^t are symmetric and $AD^t - BC^t = I$. Prove that $A^tD - C^tB = I$.

(Putnam 1986 B6) (Yufei Zhao LA P4)

9. The game of *Greed* starts with an initial configuration of one or more piles of stones. Player 1 and Player 2 take turns to remove stones, beginning with Player 1. At each turn, a player has two choices:

   - take one stone from any one of the piles (a simple move);
   - take one stone from each of the remaining piles (a greedy move).

   The player who takes the last stone wins.
   Consider the following two initial configurations:

   (a) There are 2018 piles, with either 20 or 18 stones in each pile.

   (b) There are four piles, with 17, 18, 19, and 20 stones, respectively.

   In each case, find an appropriate strategy that guarantees victory to one of the players.

(IrMO 2018, P2 Q10)

Problem of the Week

If you roll three 6-sided dice what is the expected value of the highest number rolled?