Problem Solving 2018/19

Week 7

1. Find all possible pairs of digits x and y such that 36 divides the 4-digit number with digits $x95y$.

(UCC Maths Enrichment)

2. Find the coefficient of X^5 in the polynomial $(1 + X + X^2 + X^3 + X^4 + X^5)^4$.

(UCC Maths Enrichment)

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y$$

for all x and $y \in \mathbb{R}$.

4. A rectangular array of positive integers has four rows. The sum of the entries in each column is 20. Within each row, all entries are distinct. What is the maximum possible number of columns?

(IrMO 2016, P2 Q2)

5. Let $p(x)$ be a polynomial with real coefficients. Calculate

$$\int_0^{2\pi} p(\cos x)dx.$$

6. Let M be the midpoint of side BC of an equilateral triangle ABC. The point D is on CA extended such that A is between D and C. The point E is on AB extended such that B is between A and E, and $|MD| = |ME|$. The point F is the intersection of MD and AB. Prove that $\angle BFM = \angle BME$.

(IrMO 2018 P2 Q8)

7. Let $P(x)$ be a polynomial with positive coefficients. Prove that if

$$P\left(\frac{1}{x}\right) \geq \frac{1}{P(x)}$$

holds for $x = 1$, then it holds for all $x > 0$.

(Yufei Zhao)

8. Let $A = (a_{ij})$ be a real $n \times n$ matrix satisfying

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

for all $1 \leq i \leq n$. Prove that A is invertible.

(Yufei Zhao LA P7)

9. For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n > 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

(IMO 2016 SL N1)

Problem of the Week

2018 green baubles and 2019 red baubles are initially on a Christmas tree. Rowan and William have an unlimited supply of green and red baubles and take turns in the following game. On a player's turn, they can either,

- remove any nonzero number of baubles of the same colour from the tree, or
- remove any nonzero number of baubles of one colour from the tree and replace them by the same number of baubles of the other colour.

Whoever removes the last bauble wins. Assuming Rowan takes the first turn, which player has a winning strategy?