Problem Solving 2018/19

Week 6

Problem of the Week

Suppose you want to write numbers (that are not necessarily distinct) from the set $\{1, \ldots, n\}$ on the vertices of a cube. What is the smallest n for which you can find a way to do this so that the sums along each edge are different?

(Based on P7)

1. Prove that, for $a, b, c > 0$,

   (a) $(a + b + c) \geq \sqrt[3]{abc}$.

   (b) $(a + b + c)(ab + bc + ca) \geq 9abc$.

(UCC Maths Enrichment)

2. How many strings can be formed by ordering the letters $ABCDEFGH$ in such a way that

   (a) the substring AC does not appear?

   (b) neither of the substrings AC or EG appear?

(UCC Maths Enrichment)

3. Find the smallest positive integer m such that $5m$ is an exact 5^{th} power, $6m$ is an exact 6^{th} power and $7m$ is an exact 7^{th} power.

(IrMO 2013, P1 Q1)

4. Let A and B be $n \times n$ matrices satisfying $A + B = AB$. Show that $AB = BA$.

(Yufei Zhao LA P3)

5. Find all functions $f(x) = ax^2 + bx + c$, with $a \neq 0$, such that

$$f(f(1)) = f(f(0)) = f(f(-1)).$$

(IrMO 2018 P1 Q3)

6. The sequence of positive integers $a_1, a_2, a_3, \ldots$ satisfies

$$a_{n+1} = a_n^2 + 2018 \quad \text{for } n \geq 1.$$

Prove that there exists at most one n for which a_n is the cube of an integer.

(IrMO 2018 P2 Q9)

7. Is it possible to number the 8 vertices of a cube from 1 to 8 in such a way that the value of the sum on every edge is different?

(Flanders Math Olympiad 2002 P1)

8. Determine all pairs $P(x), Q(x)$ of complex polynomials with leading coefficient 1 such that $P(x)$ divides $Q(x)^2 + 1$ and $Q(x)$ divides $P(x)^2 + 1$.

(IMC 2018 Problem 9)

9. For $R > 1$ let $\mathcal{D}_R = \{(a, b) \in \mathbb{Z}^2 : 0 < a^2 + b^2 < R\}$. Compute

$$\lim_{R\to\infty} \sum_{(a,b)\in\mathcal{D}_R} \frac{(-1)^{a+b}}{a^2 + b^2}.$$

(IMC 2018 Problem 10)