Problem Solving 2018/19

Week 5

1. Consider the 6 vertices of a regular hexagon. Each pair of vertices is connected by either a red line, or a blue line. Prove that there is a monochromatic triangle (all edges are the same colour) with three of the vertices of hexagon being its vertices.

2. Prove that the sequence defined by

$$a_0 = 2$$
$$a_n = \begin{cases} 5a_{n-1} + 3 & \text{when } n \text{ is odd} \\ 3a_{n-1} + 5 & \text{when } n \text{ is even} \end{cases}$$

contains no perfect squares.

3. Consider a game played on a 4×4 board of unit squares. Amy and Ben take turns placing stones with Amy placing the first stone. On her turn, Amy places a red stone on an unoccupied square such that no two red stones are on squares with a distance of $\sqrt{5}$ between their centres (i.e. 2 steps in a direction, 1 step perpendicular to the first direction; or a knight's move, in chess). On his turn, Ben places a blue stone on any unoccupied square. (A square occupied by a blue stone is allowed to be at any distance from any other occupied square.) They stop as soon as a player cannot place a stone.
Find the greatest K such that Amy can ensure that she places at least K red stones, no matter how Ben places his blue stones.

(Part of IMO 2018 P1)

4. A sequence of primes an is defined as follows: $a_1 = 2$, and, for all $n \geq 2$, an is the largest prime divisor of $a_1 a_2 \cdots a_{n-1} + 1$. Prove that $a_n \neq 5$ for all n.

(IrMO 1990 P1 Q2)

5. The triangle ABC is right-angled at A. Its incentre is I, and H is the foot of the perpendicular from I on AB. The perpendicular from H on BC meets BC at E, and it meets the bisector of $\angle ABC$ at D. The perpendicular from A on BC meets BC at F. Prove that $\angle EFD = 45°$.

(IrMO 2018 P1 Q2)

6. Let a, b, c be the side lengths of a triangle. Prove that

$$2(a^3 + b^3 + c^3) < (a + b + c)(a^2 + b^2 + c^2) \leq 3(a^3 + b^3 + c^3).$$

(IrMO 2018 P2 Q7)

7. Do there exist square matrices A and B such that $AB - BA = I$?

(Yufei Zhao LA P5)

8. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers such that $a_0 = 0$ and

$$a_{n+1}^3 = a_n^2 - 8 \quad \text{for} \quad n = 0, 1, 2, \ldots$$

Prove that the following series is convergent:

$$\sum_{n=0}^{\infty} |a_{n+1} - a_n|.$$

(IMC 2018 Problem 7)

9. Let $\Omega = \{(x, y, z) \in \mathbb{Z}^3 : y + 1 \geqslant x \geqslant y \geqslant z \geqslant 0\}$. A frog moves along the points of Ω by jumps of length 1. For every positive integer n, determine the number of paths the frog can take to reach (n, n, n) starting from $(0, 0, 0)$ in exactly $3n$ jumps.

(IMC 2018 Problem 8)

Problem of the Week

Fifteen stones are placed on a 4×4 board, one in each cell, the remaining cell being empty. Whenever two stones are on neighbouring cells (having a common side), one may jump over the other to the opposite neighbouring cell, provided this cell is empty. The stone jumped over is removed from the board.
For which initial positions of the empty cell is it possible to end up with exactly one stone on the board?

(Baltic Way 2018 P6)