Problem Solving 2018/19

Week 4

Monday. Every year, Vlad and Pete have to decide who teaches Linear Algebra I. To do this, they randomly generate a 2×2 matrix A with entries ± 1. (For any entry in the matrix, the probability that it is 1 is 1/2). If the matrix is invertible, Vlad teaches the module. If not, Pete teaches it. What is the probability, in a given year, that Vlad teaches Linear Algebra I? (Note that $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ being invertible is equivalent to $ad - bc \neq 0$.)

Tuesday. Each bus ticket in St Petersburg has a four digit number, from 0000 to 9999, written on it. A ticket is called *lucky* if the sum of the first two digits is equal to the sum of the last two digits. Of the 10,000 tickets, how many tickets are lucky?

Wednesday. A magician has cards numbered from 1 to 100, distributed in 3 boxes of different colors so that no box is empty. His trick consists of letting one person in the crowd choose two cards from different boxes without the magician watching. Then the person tells the magician the sum of the numbers on the two cards and he has to guess from which box no card was taken. Find a way the magician can place the cards in the boxes so that this trick always works. Find a second way of placing the cards so that the trick still works.

Thursday. You are given a knife and asked to make n straight, vertical cuts of a cake, with each cut passing the entire way through the cake(vertically and horizontally). What is the maximum number of pieces you can cut the cake into? (When $n = 2$, the answer would be 4.)

Friday. 100 mathematicians are at a party. Each of them has left their hat in the cloak-room. When the party finishes, the mathematicians form an orderly line to collect their hats. The first person enters the room and takes a random hat. After this, each mathematician takes their hat if it has not already been taken, and takes a random hat if their own hat has already been taken. What is the probability that the last mathematician takes their own hat?

1. Let S be the set of all numbers of the form $a(n) = n^2 + n + 1$, where n is a natural number. Prove that the product $a(n)a(n+1)$ is in S for all natural numbers n. Give, with proof, an example of a pair of elements $s, t \in S$ such that $st \notin S$.

(IrMO 2000 P1 Q1)

2. Consider a 9×9 chessboard. It can be covered by 27 3×1 dominoes. If we remove three of the corners, can we cover it with 26 dominoes?

3. Mary and Pat play the following number game. Mary picks an initial integer greater than 2017. She then multiplies this number by 2017 and adds 2 to the result. Pat will add 2019 to this new number and it will again be Marys turn. Both players will continue to take alternating turns. Mary will always multiply the current number by 2017 and add 2 to the result when it is her turn. Pat will always add 2019 to the current number when it is his turn. Pat wins if any of the numbers obtained by either player is divisible by 2018. Mary wants to prevent Pat from winning the game. Determine, with proof, the smallest initial integer Mary could choose in order to achieve this.

(IrMO 2018 P1 Q1)

4. Let p and q be prime numbers with $p < q$. Suppose that in a convex polygon $P_1, P_2, , P_{pq}$ all angles are equal and the side lengths are distinct positive integers. Prove that

$$P_1P_2 + P_2P_3 + \cdots + P_kP_{k+1} \geqslant \frac{k^3 + k}{2}$$

holds for every integer k with $1 \leqslant k \leqslant p$.

(IMC 2018 Problem 5)

5. Let k be a positive integer. Find the smallest positive integer n for which there exists k nonzero vectors $v_1, v_2, , v_k$ in $\mathbb{R}^n$ such that for every pair i, j of indices with $|i - j| > 1$ the vectors v_i and v_j are orthogonal.

(IMC 2018 Problem 6)