Problem Solving 2018/19

Week 3

1. Find the number of subsets of the set $\{1, 2, 3, ..., 5n\}$ such that the sum of the elements in each subset are divisible by 5.

   (Mongolia 2007 P1 Q1)

2. A natural number n is called **good** if it can be written in a unique way simultaneously as the sum $a_1 + a_2 + \cdots + a_k$ and as the product $a_1 a_2 \cdots a_k$ of some $k \geq 2$ natural numbers $a_1, a_2, \ldots, a_k$. (For example 10 is good because $10 = 5 + 2 + 1 + 1 + 1 = 5 \cdot 2 \cdot 1 \cdot 1 \cdot 1$ and these expressions are unique.) Determine, in terms of prime numbers, which natural numbers are good.

   (IrMO 1993 P1 Q2)

3. Show that given 13 points in the plane with integer coordinates, there are three of them whose center of gravity has integer coordinates.

   (Problem-Solving Methods in Combinatorics 2.1)

4. Find all real-valued functions f satisfying

$$f(2x + f(y)) + f(f(y)) = 4x + 8y$$

   for all real numbers x and y.

   (IrMO 2018 P2 Q6)

5. Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

   holds for all x, y?

   (Putnam 2003 B1)

6. Find all pairs (a, b) of natural numbers such that

$$\frac{a^2(b-a)}{b+a}$$

is a square of a prime number.

(Bosnia and Herzegovina Mathematical Olympiad 2009 P1 Q2)

7. Among the n inhabitants of an island, where n is even, every two are either friends or enemies. Some day, the chief of the island orders that each inhabitant (including himself) makes and wears a necklace consisting of marbles, in such a way that two necklaces have a marble of the same type if and only if their owners are friends.

   (a) Show that the chiefs order can be achieved by using $\frac{n^2}{4}$ different types of stones.

   (b) Prove that this is not necessarily true with less than $\frac{n^2}{4}$ types.

(Norway 2004 Q4)

8. Determine all rational numbers a for which the matrix

$$\begin{pmatrix} a & -a & -1 & 0 \\ a & -a & 0 & -1 \\ 1 & 0 & a & -a \\ 0 & 1 & a & -a \end{pmatrix}$$

is the square of a matrix with all rational entries.

(IMC 2018 Problem 3)

9. Find all differentiable functions $f : (0, \infty) \to \mathbb{R}$ such that

$$f(b) - f(a) = (b-a)f'(\sqrt{ab}) \qquad \text{for all} \qquad a, b > 0.$$

(IMC 2018 Problem 4)

Problem of the Week

There are n lamps arranged on a table. Initially all lamps are off. Two people play the following game. In each move the player flips the switch of one lamp, but must never get back an arrangement of the lit lamps that has already been on the table (including the initial arrangement). The players alternate moves, with player A moving first. The first player unable to make a move loses. Which player has a winning strategy?

(Baltic Way 2012 P6)