Problem Solving 2018/19

Week 2

1. A quadrilateral $ABCD$ is inscribed in a square with each vertex A, B, C, D on a different side of the square, and in clockwise order. Prove that

$$2 \leq |AB|^2 + |BC|^2 + |CD|^2 + |DA|^2 \leq 4.$$

(IrMO 1989 P1 Q1)

2. Let A be a nonempty set with n elements. Find the number of ways of choosing a pair of subsets (B, C) of A such that B is a nonempty subset of C.

(IrMO 1992 P1 Q3)

3. Suppose that $p_1 < p_2 < \cdots < p_{15}$ are prime numbers in arithmetic progression, with common difference d. Prove that d is divisible by 2, 3, 5, 7, 11 and 13.

(IrMO 1990 P2 Q2)

4. Let a, b, c be positive reals such that $abc = 1$. Prove that

$$a + b + c \leq a^2 + b^2 + c^2.$$

(Yufei Zhao)

5. Given an $1 \times n$ table $(n \geq 2)$, two players alternate the moves in which they write the numbers 0 and 1 in the cells of the table. The first player always writes 0, while the second always writes 1. It is not allowed for two equal numbers to appear in adjacent cells. The player who cant make a move loses the game. Which of the players has a winning strategy?

(Bosnia and Herzegovina Mathematical Olympiad 2009 P2 Q4)

6. The tangent lines from a point P meet a circle k at A and B. Let X be an arbitrary point on the shorter arc AB, and C and D be the orthogonal projections of P onto the lines AX and BX, respectively. Prove that the line CD passes through a fixed point Y as X moves along the arc AB.

(Slovenian National Mathematical Olympiad 2005 Q4.3)

7. Let n be a positive integer. In how many ways can an $n \times n$ table be filled with integers from 0 to 5 such that

   (a) the sum of each row is divisible by 2 and the sum of each column is divisible by 3;

   (b) the sum of each row is divisible by 2, the sum of each column is divisible by 3 and the sum of each of the two diagonals is divisible by 6?

(Estonia 16/17 IMO Team Selection Test S7)

8. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of positive numbers. Show that the following statements are equivalent:

   (a) There is a sequence $(c_n)_{n=1}^{\infty}$ of positive numbers such that $\sum_{n=1}^{\infty} \frac{a_n}{c_n}$ and $\sum_{n=1}^{\infty} \frac{c_n}{b_n}$ both converge;

   (b) $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ converges.

(IMC 2018 Problem 1)

9. Does there exist a field such that its multiplicative group is isomorphic to its additive group?

(IMC 2018 Problem 2)

Problem of the Week

Let $0 < x < 1$. Compute the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{\lfloor 2^n x \rfloor}}{2^n}.$$

Sources

IrMO: Irish Mathematical Olympiad
Estonia: These problems are taken from a collection of problems from Estonian olympiads, found at http://www.math.olympiaadid.ut.ee/eng/html/index.php
Problem-Solving Methods in Combinatorics: A book, written by Pablo Sóberon Bravo.
Various National Olympiad Problems were found at www.imomath.com
Yufei Zhao: A set of resources provided by Yufei Zhao.
IMC: International Mathematics Competition for University Students.
Most other problems have been found at artofproblemsolving.com