Problem Solving 2018/19

Week 1

1. The real numbers α, β satisfy the equations

$$\alpha^3 - 3\alpha^2 + 5\alpha - 17 = 0$$
$$\beta^3 - 3\beta^3 + 5\beta + 11 = 0$$

Find $\alpha + \beta$.

(IrMO 1993 P1 Q1)

2. Find the smallest prime number p for which the number

$$p^3 + 2p^2 + p$$

has exactly 42 divisors.

(Slovenian National Mathematical Olympiad 2005 Q3.2)

3. A, B, C, D are the vertices of a square, and P is a point on the arc CD of its circumcircle. Prove that

$$|PA|^2 - |PB|^2 = |PB||PD| - |PA||PC|$$

(IrMO 1988 P1 Q2)

4. Show that given a subset of $n+1$ elements of $\{1, 2, 3, ..., 2n\}$, there are two elements in that subset such that one is divisible by the other.

(Problem-Solving Methods in Combinatorics, 2.6)

5. Consider an $n \times m$ grid of squares, where n and m are positive integers. What is the largest number of points that can be marked in the vertices of the squares of the grid in such a way that no three of the marked points lie in the vertices of any right-angled triangle?

(Estonia 16/17 O18)

6. Let A be a 2×2 matrix with integer entries. Prove that if there exists a positive integer n, with $\gcd(n, 6) = 1$, such that $A^n = I$, then $A = I$.

7. Do there exist two positive powers of 5 such that the number obtained by writing one after the other is also a power of 5?

(Estonia 16/17 S1 - 2017 IMO Team Selection Test)

8. Let $a_0, a_1, \ldots, a_{n-1}$ be real numbers, where $n \geq 1$, and let the polynomial

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

be such that $|f(0)| = f(1)$ and each root α of f is real and satisfies $0 < \alpha < 1$. Prove that the product of the roots does not exceed $\frac{1}{2}^n$.

(IrMO 1993 P1 Q4)

9. Let $f : [a, b] \to [c, d]$ a strictly increasing and bijective function $(a, c > 0)$. Show that there is an unique number $x \in (a, b)$ that satisfy $\int_a^b f(t)dt = (x - a) \cdot c + (b - x) \cdot d$

Problem of the Week

Prove that every infinite arithmetic sequence $a, a + d, a + 2d, \ldots$, where $a, d \in \mathbb{N}$, contains an infinite geometric subsequence $b, bq, bq^2, \ldots$, where $b, q \in \mathbb{N}$.

(Bosnia and Herzegovina Mathematical Olympiad 2006 P2 Q4)

Sources

IrMO: Irish Mathematical Olympiad
Estonia: These problems are taken from a collection of problems from Estonian olympiads, found at http://www.math.olympiaadid.ut.ee/eng/html/index.php
Various National Olympiad Problems were found at www.imomath.com
Problem-Solving Methods in Combinatorics: Taken from this book, written by Pablo Sóberon Bravo.
Most other problems have been found at artofproblemsolving.com