

# PROBLEM SOLVING 2018/19

## *Intro 1*

1. Let  $C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$  and let  $S = \{4, 5, 9, 14, 23, 37\}$ . Find two sets  $A$  and  $B$  with the properties

- $A \cap B = \emptyset$
- $A \cup B = C$
- The sum of two distinct elements of  $A$  is not in  $S$ .
- The sum of two distinct elements of  $B$  is not in  $S$ .

(IrMO 2012, P1 Q1)

2. Prove that

$$\sqrt{ab} \leq \frac{a+b}{2}$$

whenever  $a, b \geq 0$ .

3. A number is **palindromic** if it is unchanged when its digits are reversed. Find a palindromic three digit number,  $a$ , such that  $a + 2016$  is also palindromic.
4. Suppose an ant starts on the bottom left corner of an  $8 \times 8$  chessboard. This ant can only move up and to the right. How many different paths can the ant take supposing it finishes in the top right corner of the board?
5. If  $a + b = 3$  and  $a^2 + b^2 = 7$ , what is  $a^4 + b^4$ ?
6. In a room with  $n$  people, some pairs of people know each other, and some do not. Prove that there are two people in the room who know the same number of people in the room.
7. Find a positive integer  $n$  which has remainder 2 when divided by 7, remainder 2 when divided by 6, and remainder 6 when divided by 10.
8. Show that given 5 points in the plane with integer coordinates, there are two of them whose center of gravity has integer coordinates.
9. Find all functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(x + yf(x)) = f(xf(y)) + f(y + x)$$

for all  $x$  and  $y \in \mathbb{R}$ .