Lucky bus tickets and Riemann integration

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Introduction

We are going to discuss bus tickets with serial number consisting of six digits. Any combination from 000000 to 999999 is possible.

Introduction

In the former USSR, those tickets were dispensed from machine like this one:

A ticket was considered “lucky” if the sum of the first three digits in the serial number was equal to the sum of the last three.

Introduction

When the “Lucky Bus Ticket Problem” first emerged, the computers were only a bit more advanced than the ticket dispensers, so to determine the number of lucky tickets was harder than it would be now, when a short piece of code in `Pari/GP` language

```
N=0;
for(i1=0,9,for(i2=0,9,for(i3=0,9,
for(i4=0,9,for(i5=0,9,for(i6=0,9,
if(i1+i2+i3==i4+i5+i6,N=N+1)
))))));
print(N);
```

does the trick in a fraction of a second, giving the answer 55252.

My goal today is to explain some interesting related mathematics (rather than specifically focus on computing that number).

Lucky tickets and... polynomials

Let us form a polynomial $f(x) = 1 + x + x^2 + x^3 + \cdots + x^9$.
Terms in it are just all possible digits.

What about the polynomial $f(x) \cdot f(x)$? It is a polynomial of degree 18, and the coefficient of x^k is equal to the number of sequences (a_1, a_2) of two digits with the sum k.

Similarly, for the polynomial $f(x)^3 = f(x) \cdot f(x) \cdot f(x)$, the coefficient of x^k is equal to the number of sequences (a_1, a_2, a_3) of three digits with the sum k.

Lucky tickets and... polynomials

How to encode lucky tickets, i.e. pairs of sequences with the same sum?

Consider the "Laurent polynomial" $f(x)^3 \cdot f(1/x)^3$. We already know that the coefficient of x^k in $f(x)^3$ is equal to the number of sequences (a_1, a_2, a_3) of three digits with the sum k. For the same reason, the coefficient of $1/x^k$ in $f(1/x)^3$ is equal to the number of sequences (b_1, b_2, b_3) of three digits with the sum k.

Therefore, the number of lucky tickets is the constant term of $f(x)^3 \cdot f(1/x)^3$, as now we have pairs of sequences!

Lucky tickets and... complex numbers

A very nice way to extract constant terms of Laurent polynomials uses complex numbers and trigonometry.

Let $x = e^{i\phi} = \cos\phi + i\sin\phi$, so that $x^k = e^{ik\phi}$.

Observation: the constant term of a Laurent polynomial $p(x)$ is equal to

$$\frac{1}{2\pi}\int_0^{2\pi} p(e^{i\phi})\,d\phi.$$

Indeed, integrating $\cos(k\phi)$ and $\sin(k\phi)$ for $k > 0$ gives us zero by a direct inspection, and integrating 1 gives 2π, so we just have to divide the integral by 2π to obtain the constant term.

Let us apply that general observation to our situation. We have

$$f(x) = 1 + x + x^2 + x^3 + \cdots + x^9 = \frac{1 - x^{10}}{1 - x},$$

so

$$f(x)^3 \cdot f(1/x)^3 = \left(\frac{1 - x^{10}}{1 - x}\right)^3 \cdot \left(\frac{1 - 1/x^{10}}{1 - 1/x}\right)^3 = \left(\frac{2 - x^{10} - 1/x^{10}}{2 - x - 1/x}\right)^3$$

Therefore, substituting $x = e^{i\phi}$, we get

$$f(e^{i\phi})^3 \cdot f(1/e^{i\phi})^3 = \left(\frac{2 - e^{10i\phi} - e^{-10i\phi}}{2 - e^{i\phi} - e^{-i\phi}}\right)^3 = \left(\frac{2 - 2\cos 10\phi}{2 - 2\cos\phi}\right)^3$$

Simplifying further, we have

$$f(e^{i\phi})^3 \cdot f(1/e^{i\phi})^3 = \left(\frac{4(\sin 5\phi)^2}{4(\sin \frac{\phi}{2})^2}\right)^3 = \left(\frac{\sin 5\phi}{\sin \frac{\phi}{2}}\right)^6 .$$

Remembering our formula with an integral, we get the following formula for the number of lucky tickets:

$$\frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\sin 5\phi}{\sin \frac{\phi}{2}}\right)^6 \, d\phi,$$

or, letting $\phi = 2\theta$,

$$\frac{1}{\pi}\int_0^{\pi} \left(\frac{\sin 10\theta}{\sin \theta}\right)^6 \, d\theta.$$

Lucky tickets and... everything

A peculiar consequence is that the number

$$\frac{1}{\pi}\int_0^{\pi}\left(\frac{\sin 10\theta}{\sin\theta}\right)^6 d\theta.$$

is actually an integer.

A very similar argument shows that the number of lucky bus tickets with a serial number of length $2n$ using base m digits is equal to

$$\frac{1}{\pi}\int_0^{\pi}\left(\frac{\sin m\theta}{\sin\theta}\right)^{2n} d\theta.$$

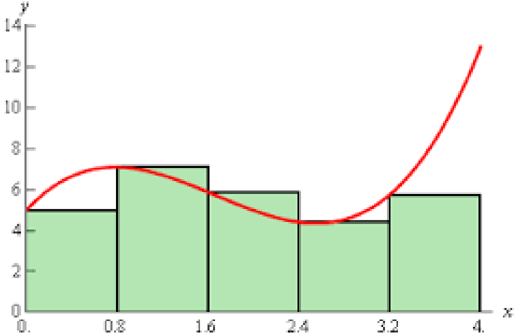
We shall now discuss how to compute integrals like that.

Lucky tickets and... Riemann sums

Integrals are defined as limits of Riemann sums: $\int_a^b f(x)\,dx$ is computed as the limit of sums

$$\sum_{j=1}^{M} f(x_j)\Delta x_j,$$

as the following picture suggests:



Riemann sums of Laurent polynomials

Suppose that we have a Laurent polynomial $p(x) = \sum_{k=-N}^{N} a_k x^k$, so that

$$p(e^{i\phi}) = \sum_{k=-N}^{N} a_k e^{ik\phi}.$$

We already know that $\int_0^{2\pi} p(e^{i\phi})\, d\phi = 2\pi a_0$. Let us look at a Riemann sum with the nodes $\phi_j = \frac{2\pi j}{M}$, where $j = 1, \ldots, M$.

$$\sum_{j=1}^{M} p(e^{i\phi_j})\Delta\phi_j = \sum_{j=1}^{M} \sum_{k=-N}^{N} a_k e^{ik\phi_j} \Delta\phi_j =$$

$$\sum_{j=1}^{M} \sum_{k=-N}^{N} a_k e^{ik\frac{2\pi j}{M}} \frac{2\pi}{M} = \frac{2\pi}{M} \sum_{k=-N}^{N} a_k \sum_{j=1}^{M} \left(e^{\frac{2\pi ik}{M}}\right)^j$$

Riemann sums of Laurent polynomials

Let us note that

$$\sum_{j=1}^{M} \left(e^{\frac{2\pi ik}{M}}\right)^j = \begin{cases} M, & \text{if } k \text{ is divisible by } M, \\ 0, & \text{otherwise .} \end{cases}$$

Indeed, if k is divisible by M, we add 1 to itself M times. Otherwise, $e^{\frac{2\pi ik}{M}} \neq 1$, so the geometric series applies.

Therefore, our Riemann sum

$$\frac{2\pi}{M} \sum_{k=-N}^{N} a_k \sum_{j=1}^{M} \left(e^{\frac{2\pi ik}{M}}\right)^j$$

is given by

$$\frac{2\pi}{M} \sum_{\substack{k=-N,\ldots,N \\ k \text{ divisible by } M}} a_k M = 2\pi \sum_{\substack{k=-N,\ldots,N \\ k \text{ divisible by } M}} a_k .$$

Riemann sums of Laurent polynomials

A fun conclusion is that if we take $M > N$, then the only k between $-N$ and N that is divisible by M is $k = 0$, so our Riemann sum

$$\frac{2\pi}{M} \sum_{k=-N}^{N} a_k \sum_{j=1}^{M} \left(e^{\frac{2\pi i k}{M}}\right)^j$$

is equal to

$$2\pi \sum_{\substack{k=-N,\dots,N \\ k \text{ divisible by } M}} a_k = 2\pi a_0 = \int_0^{2\pi} p(e^{i\phi})\, d\phi.$$

Conclusion: equidistant Riemann sums for *trigonometric polynomials* with the number of nodes bigger than the degree of the polynomial give precise value of the integral.

For example, for the original problem we need to take the Riemann sum with 28 nodes, since our polynomial is of degree 27.

A silly example

I shall leave it to you as an exercise to do that calculation and to obtain the number 55252 as a result, and instead do the very trivial example of two-digit tickets. Clearly, the number of lucky two-digit tickets is equal to 10. However, we can also express it as an integral

$$\frac{1}{\pi}\int_0^{\pi}\left(\frac{\sin 10\theta}{\sin\theta}\right)^2\,d\theta,$$

which arises from a trigonometric polynomial of degree 9. Therefore, ten nodes would do. Let us take the equidistant nodes $\frac{\pi}{20}, \frac{3\pi}{20}, \frac{5\pi}{20}, \ldots, \frac{19\pi}{20}$. (Since we integrate over a period, does not matter where we start.)

We know that the integral is equal to the Riemann sum, so that

$$\frac{1}{\pi}\int_0^{\pi}\left(\frac{\sin 10\theta}{\sin\theta}\right)^2\,d\theta = \frac{1}{\pi}\frac{2\pi}{20}\sum_{i=1}^{10}\left(\frac{\sin 10\theta_i}{\sin\theta_i}\right)^2,$$

A silly example

To compute

$$\frac{1}{\pi}\frac{2\pi}{20}\sum_{i=1}^{10}\left(\frac{\sin 10\theta_i}{\sin\theta_i}\right)^2,$$

we note that $\sin 10\theta_i = \pm 1$ for all our nodes θ_i, so we obtain

$$\frac{1}{10}\left(\frac{1}{\left(\sin\frac{\pi}{20}\right)^2}+\frac{1}{\left(\sin\frac{3\pi}{20}\right)^2}+\frac{1}{\left(\sin\frac{5\pi}{20}\right)^2}+\frac{1}{\left(\sin\frac{7\pi}{20}\right)^2}+\frac{1}{\left(\sin\frac{9\pi}{20}\right)^2}+\right.$$

$$\left.\frac{1}{\left(\sin\frac{11\pi}{20}\right)^2}+\frac{1}{\left(\sin\frac{13\pi}{20}\right)^2}+\frac{1}{\left(\sin\frac{15\pi}{20}\right)^2}+\frac{1}{\left(\sin\frac{17\pi}{20}\right)^2}+\frac{1}{\left(\sin\frac{19\pi}{20}\right)^2}\right)=$$

$$\frac{1}{10}\left(\frac{2}{\left(\sin\frac{\pi}{20}\right)^2}+\frac{2}{\left(\sin\frac{3\pi}{20}\right)^2}+\frac{2}{\left(\sin\frac{5\pi}{20}\right)^2}+\frac{2}{\left(\sin\frac{7\pi}{20}\right)^2}+\frac{2}{\left(\sin\frac{9\pi}{20}\right)^2}\right)$$

A silly example

Furthermore,

$$\frac{1}{10}\left(\frac{2}{\left(\sin\frac{\pi}{20}\right)^2}+\frac{2}{\left(\sin\frac{3\pi}{20}\right)^2}+\frac{2}{\left(\sin\frac{5\pi}{20}\right)^2}+\frac{2}{\left(\sin\frac{7\pi}{20}\right)^2}+\frac{2}{\left(\sin\frac{9\pi}{20}\right)^2}\right)=$$

$$\frac{1}{10}\left(\frac{2}{\left(\sin\frac{\pi}{20}\right)^2}+\frac{2}{\left(\sin\frac{3\pi}{20}\right)^2}+\frac{2}{\left(\sin\frac{5\pi}{20}\right)^2}+\frac{2}{\left(\cos\frac{3\pi}{20}\right)^2}+\frac{2}{\left(\cos\frac{\pi}{20}\right)^2}\right)=$$

$$\frac{1}{10}\left(\frac{2}{\left(\sin\frac{\pi}{20}\cos\frac{\pi}{20}\right)^2}+\frac{2}{\left(\sin\frac{3\pi}{20}\cos\frac{3\pi}{20}\right)^2}+\frac{2}{\left(\sin\frac{5\pi}{20}\right)^2}\right)=$$

$$\frac{1}{10}\left(\frac{8}{\left(\sin\frac{\pi}{10}\right)^2}+\frac{8}{\left(\sin\frac{3\pi}{10}\right)^2}+4\right).$$

A silly example

Finally, we recall that the number that we were computing here is the number of two-digit lucky tickets, which is equal to 10, so

$$10 = \frac{1}{10}\left(\frac{8}{\left(\sin\frac{\pi}{10}\right)^2} + \frac{8}{\left(\sin\frac{3\pi}{10}\right)^2} + 4\right).$$

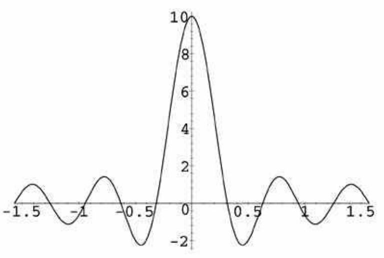
This leads to a rather unexpected trigonometric formula

$$\frac{1}{\left(\sin\frac{\pi}{10}\right)^2} + \frac{1}{\left(\sin\frac{3\pi}{10}\right)^2} = 12,$$

which would be tricky to prove otherwise!

An estimate

We may integrate over $[-\pi/2, \pi/2]$ instead of $[0, \pi]$ since the function is periodic. The graph of the function $\frac{\sin 10\theta}{\sin \theta}$ looks like this:



On $[-\pi/2, \pi/2]$, the function $f(\theta) = \frac{\sin 10\theta}{\sin \theta}$ reaches its maximal value 10 at $x = 0$, and gets much smaller towards the ends of the interval. Raising it to the power 10 magnifies this qualitative behaviour, and the *saddle point method* from [very] advanced calculus applies.

An estimate

Roughly, that methods says the following: write

$$\ln \frac{f(\theta)}{f(0)} = -q\theta^2 + o(\theta^2),$$

and then use the approximations (valid for large A)

$$\int_{-\epsilon}^{\epsilon} f(\theta)^A, d\theta = f(0)^A \int_{-\epsilon}^{\epsilon} e^{A \ln \frac{f(\theta)}{f(0)}}\, d\theta \approx$$

$$f(0)^A \int_{-\epsilon}^{\epsilon} e^{-Aq\theta^2}\, d\theta \approx f(0)^A \int_{-\infty}^{\infty} e^{-Aq\theta^2}\, d\theta = f(0)^A \frac{\sqrt{2\pi}}{Aq}.$$

For the case of lucky tickets, that is $A = 6$, one gets for that number the approximate value $\frac{10^6}{3\sqrt{11\pi}} \approx 56700$, a very good approximation! People say that Vladimir Drinfeld, a Fields medal laureate who now works in Chicago, came up with this estimate when he was a high school student!

That's all folks!

Thank you for your patience!